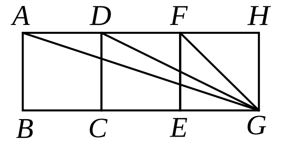
AMATYC Contest (Spring 2006) SOLUTIONS

1. [E] $f(g(1)) - g(f(1)) = f(2) - g(-1) = 1 - (-2) = 3$.
2. [A] The thousands place has to be 2 or 6, then exactly one of the remaining places is the other nonzero digit. The two places left are 0s. This gives $(2)(3) = 6$ possibilities.
3. [C] $\tan(m\angle GAH + m\angle GDH) = \dfrac{\tan(m\angle GAH) + \tan(m\angle GDH)}{1 - \tan(m\angle GAH)\cdot\tan(m\angle GDH)} = \dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1$.



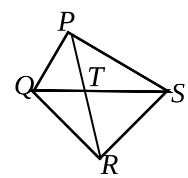
Therefore $m\angle GAH + m\angle GDH = 45°$. The answer follows.
4. [B] The cost of the first horse is $\$200/80\% = \$250$, and the loss of selling is \$50. Thus the profit from selling the second horse is also \$50, therefore the cost of the second horse is $\$50/25\% = \$200$. The answer is $\$250 + \$200 = \$450$.
5. [C] $A(17,49) = \{17,18,\cdots,65\}$, $A(49,17) = \{49,50,\cdots,65\}$. The answer follows.
6. [A] $M = \sum_{n=1}^{k}[(\ln a + \ln n) - (\ln b + \ln n)] = \sum_{n=1}^{k}(\ln a - \ln b) = k(\ln a - \ln b) = k\ln\frac{a}{b}$.
$N = e^{k\ln(a/b)} = (e^{\ln(a/b)})^k = (\frac{a}{b})^k$. The answer follows.
7. [B] The key is that 5 and 7 are coprime, so are 5 and 8, whereas 6, 8 are not coprime. So, if I is true, then $(x^5)^3(x^7)^{-2} = x$ is rational. Likewise, if III is true, then $(x^5)^5(x^8)^{-3} = x$ is rational. Whereas, with $x = \sqrt{2}$, II would be true while x is not rational.
8. [B] Let A_k be the set of all positive integers less than 1000 that are divisible by k. Let $|S|$ stand for the number of elements in set S. We need $|A_3| - |A_3 \cap A_2| - |A_3 \cap A_9| + |A_3 \cap A_2 \cap A_9|$
$= |A_3| - |A_6| - |A_9| + |A_{18}| = 333 - 166 - 111 + 55 = 111$. So the answer is $\frac{111}{999} = \frac{1}{9}$.
9. [B] Since r and s are the two solutions to $x^2 + 3x + c = 0$, we have $r + s = -3$ and $rs = c$. Now, $2rs = (r+s)^2 - (r^2 + s^2) = (-3)^2 - 33 = -24$. So $c = rs = -12$.
10. [E] Drop at 12 ft. If it breaks, then, with 11 trials remaining, drop at 1ft, then 2 ft, then 3ft, etc. until it breaks or until the trials run out. Else (if dropping at 12 ft didn't break the ball) jump to drop at $12 + 11 = 23$ ft. If it breaks, then, with 10 trails remaining, drop at $12 + 1 = 13$ ft, $12 + 2 = 14$ ft, $12 + 3 = 15$ ft, etc. until it breaks or until the trials run out. Else (if dropping at $12 + 11 = 23$ ft didn't break the ball) jump to drop at $12 + 11 + 10 = 33$ ft. If it breaks, then, with 9 trials remaining, drop at $12 + 11 + 1 = 24$ ft, $12 + 11 + 2 = 25$ ft, $12 + 11 + 3 = 26$ ft, etc. until it breaks or until the trials run out. Continue this way. Such a strategy can determine with certainty the greatest whole number of feet from which a ball can be dropped without breaking provided it is no greater than $12 + 11 + 10 + 9 + \cdots + 3 + 2 = (12 + 2)(11)/2 = 77$.
11. [A] Let B be the point on $\overline{MT}$ such that $\overline{CB} \perp \overline{MT}$. Then $CB = 63$, $MB = 16$. The Pythagorean Theorem gives $CM = \sqrt{63^2 + 16^2} = 65$. Then ΔAMC is a 5:4:3 right triangle (as 65:52:39 is 5:4:3.) The area of the pentagon is the area of the trapezoid $CMTY$ plus the area of the right triangle ΔAMC. This gives $\frac{1}{2}(63 + 79)(63) + \frac{1}{2}(52)(39) = 5487$.
12. [A] Let the increasing nonnegative integers be x_1, x_2, x_3, x_4, x_5, x_6. Now, 5 is halfway between x_3 and x_4 as the median is 5. Likewise, 5 is halfway between x_1 and x_6 as the midrange is 5. The mean is also 5, so 5 has to be halfway between x_2 and x_5. Thus we only have to enumerate the possible values of x_1, x_2, x_3. They are from 0, 1, 2, 3, 4, and so the answer is $C_3^5 = 10$.
13. [E] First, note that $x^2 + 2x + 2$ is $(x+1)^2 + 1$ and so it is always positive. For the case $x^3 + 4x^2 - 6x - 22 = x^2 + 2x + 2$, we have $x^3 + 3x^2 - 8x - 24 = 0$, i.e. $(x+3)(x^2 - 8) = 0$. For the case $x^3 + 4x^2 - 6x - 22 = -(x^2 + 2x + 2)$, we have $x^3 + 5x^2 - 4x - 20 = 0$, i.e. $(x+5)(x^2 - 4) = 0$. So the solutions are -3, $2\sqrt{2}$, $-2\sqrt{2}$, -5, 2, -2. The sum of their absolute values is $3 + 2\sqrt{2} + 2\sqrt{2} + 5 + 2 + 2 = 12 + 4\sqrt{2}$. The answer follows.

14. [D] The sum of the three digits are from 1, 3, 5, 7, 9. Their sum has to be divisible by 9, and so can be 9, 18, 27. For a sum of 9, it has to be 711, 531, 333, or numbers resulting from rearranging the digits – there are $3+3!+1=10$ possibilities. A sum of 18 is impossible, as all three digits are odd. A sum of 27 comes from only 999. So we have a total of 11 possibilities.

15. [C] $\alpha=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$. So $\tan\alpha=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$.

16. [D] If $x\geq 0$, then $y^2-2xy+x^2=0$, i.e. $(y-x)^2=0$, so $y=x$, thus $(x,y)=(1/\sqrt{2},1/\sqrt{2})$. If $x<0$, then $y^2-x^2=0$, so $y=\pm x$, and so $(x,y)=(-1/\sqrt{2},-1/\sqrt{2})$ or $(-1/\sqrt{2},1/\sqrt{2})$.

17. [C] $A=(a,f(a))$. Let C be the mirror image of A with respect to the line $y=x$. Thus $C=(f(a),a)$. Then B is the midpoint of $\overline{AC}$. The vertical line through A and the horizontal line through C meet at a point D on the line $y=x$, with ΔADC being an isosceles right triangle. $AD=f(a)-a$, so $AC=(f(a)-a)\sqrt{2}$ Thus $AB=(f(a)-a)\dfrac{\sqrt{2}}{2}$.



18. [D] ΔSQR is an isosceles right triangle, ΔSQP is a $30°$-$60°$-$90°$ right triangle. Thus the points P, Q, R, S fall on a circle with $\overline{SQ}$ being a diameter. Therefore $\angle RTS=\angle TPS+\angle TSP=\angle RPS+\angle TSP=\angle RQS+\angle TSP=45°+30°=75°$.

19. [D] A composite number is not circumfactorable precisely when it is of the form p_1p_2, where p_1 and p_2 are distinct primes. To prove this, observe that such a p_1p_2 is indeed not circumfactorable.. Also observe that a composite number of the form $p_1{}^2$, $p_1{}^2p_2$ or $p_1p_2p_3$ is circumfactorable, where p_1, p_2, p_3 are distinct primes. To complete the proof, we only have to show that if m is a circumfactorable composite number then, for a prime p, the number mp is also circumfactorable. To do this, let all the factors of m that are greater than 1 be arranged around a circle so that any two adjacent factors have a common factor greater than 1. Call such an arrangement "permissible." Suppose p itself is a factor of m. A factor greater than 1 of mp must fall into one of the following two mutually exclusive cases: (1) It is already a factor of m, and so already on the circle; (2) It is not a factor of m but is of the form ap, where $a>1$ is a factor of m already on the circle; For each factor ap of mp that falls in case (2), place ap right next to a. (Either side is okay.) This results in a permissible arrangement for mp. If p itself is not a factor of m, then in addition to (1) and (2) there will also be: case (3) p. In this situation, for each factor of m originally on the circle there is exactly one factor of mp that falls into case (2). Though we can place ap on either side of a, let's assume that we make sure at least two factors a_1, a_2 of m originally adjacent on the circle have their case (2) counterparts a_1p, a_2p placed adjacent to each other. With this, we deal with the remaining factor p at the very end of the process by placing it right between a_1p and a_2p. This concludes the proof. We now count composite numbers, less than 200, of the form p_1p_2, where $p_1<p_2$ are distinct primes. Note that p_1 can only be 2, 3, 5, 7, 11, 13. For each p_1, the prime p_2 is greater than p_1 and can run up to a certain value. For example for $p_1=2$, we have p_2 being any of the 24 primes 3, 5, 7, 11, …, 97 (Of course, we have to memorize all prime numbers less than 100.) In the end, we enumerate all possibilities, and it works out to be $24+16+9+5+2+0=56$

20. [A] There are $(1003)(2004)$ possible right triangles. This is because the hypotenuse must be a diameter, which has 1003 possibilities, for each of which there are 2004 choices for the remaining vertex. There are $(2006)(1002)$ possible isosceles triangles. To see this, first observe that such an isosceles triangle cannot be equilateral because 2006 is not divisible by 3. Thus it has a distinguished vertex – the one where the two sides of equal length meet. This vertex can be any of the 2006 points. For each choice, there will be 1002 possible way to choose the opposite side. Since $(1003)(2004)=(2006)(1002)$, it follows that $I=R$.