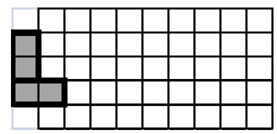
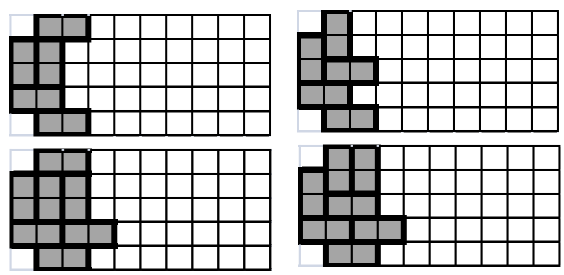
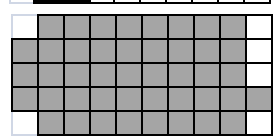
Laney College

AMATYC Contest (Spring 2011) SOLUTIONS

1. [D] $0.8x = 2(0.6)$. So $x = 1.5 = 150\%$.
2. [B] $(ab^2 + a)3^2 + (ab^2 + a) = 250$, i.e. $a(b^2 + 1) = 25$. So $a = 5$, $b^2 + 1 = 5$ ($b = 2$.)
3. [E] "442": 3, "4411": $C_2^4 = 6$, "4222": 4, "42211": $5 \cdot C_2^4 = 5 \cdot 6 = 30$, "421111": $6 \cdot 5 = 30$, "4111111": 7, "22222": 1, "222211": $C_4^6 = 15$, "2221111": $C_3^7 = 35$, "22111111": $C_2^8 = 28$, "211111111": 9, "1111111111": 1.
4. [A] Let the sum of the 3rd and the 4th be $k$. The sum of the six numbers is $3k$. So $k$ is odd, with $3k$ being a cube. The smallest possible $k$ is $3^2 = 9$ (so $n = 2$.) The next two are $k = 3^2 3^3 = 243$ (the middle two 121, 122, with $n = 119$) and $k = 3^2 5^3 = 1125$ (the middle two 562, 563, with $n = 560$.)
5. [B] The first series: $a$, $ar$, $ar^2$, …. We are told that $a/(1-r) = 6$ and $a^2/(1-r^2) = 15$. Take the quotient of the latter by the former to get $a/(1+r) = 15/6 = 2.5$.
6. [E] The first sentence implies that 8 was the 4th number originally, so 4 is the 3rd. The answer follows.
7. [B] 8 2x3s won't work: It's impossible to cover the 3-cell column. 12 1x4s won't work: The 3 cells on the far left have to be covered by 3 horizontal 1x4s, necessitating a horizontal one at the left end of both the 1st and the 5th rows; the same argument is repeated one more round, then it's clear it won't work. 16 1x3s would work: Fill the 3-cell col with a single 1x3, with the rest covered by horizontal 1x3s. Now prove that 24 1x2s won't work: Suppose it did. Among all such coverings there would be one minimizing the number of horizontal 1x2s. This covering must have no two horizontal 1x2s side by side forming a square, otherwise they could be replaced by 2 vertical 1x2s, contracting the assumption of minimization. Work our way from the far left. There should be exactly 1 horizontal 1x2 protruding from the 1st col into the 2nd col, in the 2nd or the 4th row. Then there should be 2 horizontal 1x2s protruding from the 2nd col into the 3rd col, as shown. Then there must be exactly one horizontal 1x2 protruding from the 3rd col into the 4th col, in the 4th (or the 2nd) row. Then things begin to repeat themselves, ultimately with an isolated corner cell when we first intrude into the last col--and get stuck. Thus the answer.







8. [C] The smallest integer $n$ less than $17+12$ such that $n^2 > 17^2 + 12^2 = 433$ is 21.
9. [E] The degree $\le 4$, otherwise $P(3) \ge 3^5 = 243 > 144$. So $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + 3$, with $\begin{cases} a_4 + a_3 + a_2 + a_1 = 5 \\ 16a_4 + 8a_3 + 4a_2 + 2a_1 = 36 \\ 81a_4 + 27a_3 + 9a_2 + 3a_1 = 141 \end{cases}$, i.e. $\begin{cases} a_4 + a_3 + a_2 + a_1 = 5 \\ 8a_4 + 4a_3 + 2a_2 + a_1 = 18 \\ 27a_4 + 9a_3 + 3a_2 + a_1 = 47 \end{cases}$. Thus $7a_4 + 3a_3 + a_2 = 13$ and $19a_4 + 5a_3 + a_2 = 29$. So $12a_4 + 2a_3 = 16$, i.e. $6a_4 + a_3 = 8$, and so $a_4 = 1$, $a_3 = 2$, $a_2 = 0$, $a_1 = 2$.
10. [E] We work our way from the leading digit, trying to make the number as small as possible. So we start with 11, then the next digit must be 3 to get 113. Then try 1131, and try 11311, thus 113113. Now we see the pattern, leading to 1131131131.
11. [B] Scaling both sequences if necessary, assume the arithmetic sequence as 1, $1+d$, $1+2d$, etc., the geometric sequence as $a$, $ar$, $ar^2$, etc. So $r(1+d) = 180/96$, and $r(1+2d)/(1+d) = 324/180$. Take the quotient to get $(1+2d)/(1+d)^2 = 24/25$. Solve to get $d = -1/6,\ 1/4$, with $r = 9/4,\ 3/2$. For the first case, the arithmetic sequence is

(after scaling by 6) 6, 5, 4, 3, …, and the geometric sequence is 16, 36, 81, 729/4, …, with the product sequence disagreeing with what's given at the 4th term. The second scenario works, with the arithmetic sequence (after scaling by 4) 4, 5, 6, 7, 8, …, and the geometric sequence 24, 36, 54, 81, 243/2, ….. So $(8)(243/2)=972$.

12. [B] $u=\log_x y=\log_2 y/\log_2 x$ and $1/u=\log_y x=\log_2 x/\log_2 y$. So $u+1/u=29/10$, $10u^2-29u+10=0$, $(2u-5)(5u-2)=0$, so $u=2/5, 5/2$. Given the symmetry of the problem in $x$ and $y$, we may assume $u=2/5$. Thus $\log_2 y/\log_2 x=2/5$. But $xy=128$, so $\log_2(xy)=7$, i.e. $\log_2 x+\log_2 y=7$. It follows that $\log_2 x=5$, $\log_2 y=2$, and so $x=32$, $y=4$, with $x+y=36$.

13. [C] Calculate $2011-(nonprime)^2$ for each of the nonprimes to get 567, 411, 247, 75, $-105$. So rule out E. This is the sum of $a^5+b^2$, with $a$, $b$ being primes. One of them has to be 2 in order for this to be odd. It quickly follows that $a=3$, $b=2$ would give 247. The answer is thus C.

14. [C] The palindrome $xyyx$ is $1001x+110y$. So $1001x+110y\equiv 0$ mod 17. But $1001\equiv -2$ mod 17 and $110\equiv 8$ mod 17. So $-2x+8y\equiv 0$, so $2(4y-x)\equiv 0$ mod 17. But 17 is prime, allowing us to conclude $4y\equiv x$ mod 17. So $y$ can be 1, 2, 5, 6, 9, with $x$ being 4, 8, 3, 7, 2, respectively.

15. [D] First of all, $a<b<c<d$, and $x$ has to be either between $a$ and $b$ or between $b$ and $c$, and $y$ is either between $c$ and $d$ or between $b$ and $c$. There are five possibilities for the ordering: $axbcyd$, $axbycd$, $abxcyd$, $abxycd$, $abyxcd$. Choose six from the seven numbers (7 ways) and use one of the schemes, so $7\times 5=35$.

| | | |
|---|---|---|
| $a$ | $b$ | $y$ |
| $x$ | $c$ | $d$ |

16. [C] Let $a_8=x$. So $a_7=160$, $a_6=x-160$, $a_5=320-x$, $a_4=2x-480$, $a_3=800-3x$, $a_2=5x-1280$, $a_1=2080-8x$. Demanding each of them to be positive leads to $256<x<260$, so $x$ can be 257, 258, 259. Only 259 would give an increasing sequence. Thus the answer.

17. [C] Use the idea of the Euclidian Algorithm: $n^2+7=(n+4)(n-4)+23$. So $1\neq GCF(n^2+7, n+4)=GCF(n+4, 23)$. Thus $GCF(n+4, 23)=23$, i.e. $n+4$ is divisible by 23. But $1\le n\le 2011$, so $n=23k-4$, $k=1, 2, \ldots, 87$.

18. [A] Let $P(x,y)=(2x+y)(4x+y)(6x+y))\cdots(20x+y)$. If $P_{odd}(x,y)$ is the part of $P(x,y)$ with odd powers in $y$, then what we want is

$$\frac{P_{odd}(1,1)}{P(1,1)}=\frac{P(1,1)-P(1,-1)}{2P(1,1)}=\frac{(3)(5)(7)\cdots(21)-(1)(3)(5)\cdots(19)}{2(3)(5)(7)\cdots(21)}=\frac{(3)(5)\cdots(19)(21-1)}{2(3)(5)(7)\cdots(21)}=\frac{10}{21}$$
.

19. [D] $1+2+3+\cdots+15=120$. If there were 10 sets, the common sum would be 12, with the element 15 nowhere to go. 8 sets would work: Each set has a sum of $120/8=15$, with 15 alone, 1 and 14 paired up, 2 and 13 paired up, etc.

20. [C] The four primes are separated, creating five spaces (including one to the far left and one to the far right), with $x_1$, $x_2+1$, $x_3+1$, $x_4+1$, $x_5$ integers respectively, $x_i\ge 0$, $x_1+x_2+x_3+x_4+x_5=2$. There are $C_2^{2+5-1}=C_2^6=15$ such partitions of 2, leading to the answer $\dfrac{(15)(4!)(5!)}{(9!)}=\dfrac{5}{42}$.