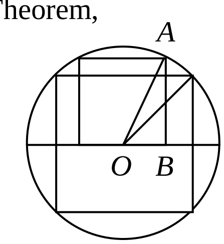
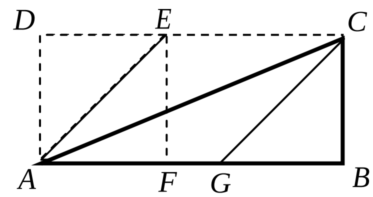
Laney College (v. 2 [1/30/2015] – Updated #4)

AMATYC Contest (Fall 2011) SOLUTIONS

1. [E] $2\cdot 3 \wedge 2+3 = 2\cdot 3\wedge 5 = 6\wedge 5 = 7776$
2. [B] $(1.20)(0.90)(7/6) = 1.26$, which represents a 26% increase.
3. [D] $2a-3b=8$, $2a+3b=20$, solve to get $a=7$, $b=2$, so $a+b=9$
4. [D] a can be 3, 2, 1, with b^2+c^2 being 1282, 1947, 2010. Prime factorize them: $1282 = 2\cdot 641$, $1947 = 3\cdot 11\cdot 59$, $2010 = 2\cdot 3\cdot 5\cdot 67$. Keep only those cases where all primes of the form $4k+3$ (such as 3) appear an even number of times (including 0 times). So only 1282 works. Any prime of the form $4k+1$ (such as 641) can be written as a sum of squares. So $2\cdot 641 = (1^2+1^2)(25^2+4^2) = (1+i)(1-i)(25+4i)(25-4i)$ $= (1+i)(25+4i)(1-i)(25-4i) = (21+29i)(21-29i) = 21^2+29^2$. The answer is thus $3+21+29=53$. (Remark: The solution uses knowledge in Gaussian integers.)
5. [C] The ratio of white to red can be 1/1, 2/1, 3/1, 4/1, 5/1, 1/2, [2/2], 3/2, [4/2], 5/2, 1/3, 2/3, [3/3], 4/3, 5/3, 1/4, [2/4], 3/4, [4/4], 5/4, along with the case of pure red and that of pure while. The cases enclosed in brackets are duplication of some ratios already listed.
6. [A] $2-2x=-2$ or 6, so $x=2$ or -2.
7. [B] $P_1(t) = P\exp(\lambda(t-t_0))$ and $P_2(t) = P\exp(-\lambda(t-t_0))$, where t_0 indicates Jan. 1, 2009. So $P_1(t)P_2(t) = P^2$ for all t.
8. [C] $b=9$, $c=6$.
9. [B] Suppose he traveled x miles in the first ½ hours. Then $x+(x-2.5)+(x-5)+(x-7.5)+(x-10)+(x-12.5)+\frac{2}{3}(x-15)=197.5$. Solve to get $x=36.75$. Then calculate $x+(x-2.5)+(x-5)+(x-7.5)$. to be 132.
10. [D] Say x liters were removed, which contains $0.80x$ liters of water. So the net effect at the end is as if the said $0.80x$ liters of water is replaced by antifreeze. So the antifreeze content increases by $0.80x$ from 2 liters to 2.5 liters. So $2+0.80x=2.5$, so $x=0.625$.
11. [E] There are more straightforward approaches. But this one is interesting: $(((((2+1)3+1)4+1)5+1)6+1)7 = 8659$. To understand this, denote by $F(k)$ the number of nonzero numbers with at most k digits that can be formed using $k+1$ available mutually distinct digits, without reuse. Then the problem aims to calculate $F(6)$. But $F(6) = (F(5)+1)7$ because there are 7 choices for the ones digit, and sitting to its left is either nothing (thus the “1”) or else a number with at most 5 digits built from the remaining 6 available digits, without reuse. The iteration continues, i.e. $F(5) = (F(4)+1)6$, …, $F(2) = (F(1)+1)3$, and obviously $F(1)=2$.
12. [D] The vector $<3,-2>$ is parallel to the line $2x+3y=24$, and the vector $<2,-3>$ is parallel to the line $3x+2y=6$. The two vectors are of the same length. So the sum $<3,-2>+<2,-3> = <5,-5>$ and the difference $<3,-2>-<2,-3> = <1,1>$ are each parallel to a line with respect to which the lines $2x+3y=24$ and $3x+2y=6$ are symmetric. Since we want the one with a positive slope, so it is $<1,1>$ that is relevant. So the line is of slope 1, passing through where $2x+3y=24$ and $3x+2y=6$ meet, namely the point $(-6,12)$. This is the line $y=x+18$. The answer follows.
13. [B] The two squares are shown in the accompanying picture. Let the radius be r. So $OA=r$, and $OB=\frac{1}{2}AB$, so $AB^2+(\frac{1}{2}AB)^2 = r^2$ by the Pythagorean Theorem, and so the area of the smaller square is $AB^2 = 4r^2/5$. The length of each side of the larger square is $r\sqrt{2}$, and so the area of that square is $2r^2$. So the ratio of the area of the smaller square to that of the larger square is $(4r^2/5)/(2r^2)$, i.e. 2/5. The answer is $(2/5)(45)=18$.





14. [C] The left edge AD is taken to AF, and DE to FE, when the bill is folded along AE. It is then folded along AC, taking CE to CG, and AE to AG. In $AGCE$, $135° = \angle AEC = \angle AGC$, and $\angle EAG = 45°$, so $\angle ECG = 45°$. So $\angle GCB = 45°$. It follows that $157\,\text{mm} = DE + CE = BC + \sqrt{2}BC = (1+\sqrt{2})BC$, so $BC = 157\,\text{mm}/(1+\sqrt{2}) \approx 65\,\text{mm}$.
15. [A] Begin with one box. Each time when a box is made nonempty by having 5 new boxes placed in it, the total number of boxes increases by 5. Once this is done to the 18^{th} such box, there are therefore a total of $1+18\cdot 5 = 91$ box, with thus $91-18 = 73$ empty boxes.
16. [D] If Al wins math, then Al is right, and so Di wins bio, thus Di is right, and so Bo wins physics, thus Bo is wrong, and so Cy doesn't win chemistry, which is impossible, as it is the only award remaining for Cy to win. This rules out Al as the math winner. Similar arguments rule out Bo and Cy as the math winner. Di winning math can be seen to lead to Bo winning physics, while Al has to be wrong (because Al thought Di would win bio) and thus Al must be winning chemistry, leaving Cy to win biology. And this winning pattern can be easily checked to work.
17. [C] We easily rule out prime numbers 71 and 73. And 74 has the prime factorization $2\cdot 37$, and so cannot be the product of a group. For 70, the group must be {2, 5, 7}. The products of the products of the other two groups equals $1\cdot 3\cdot 4\cdot 6\cdot 8\cdot 9 = 72^2$, thus one group has to have a product at least 72, exceeding 70, a contradiction. So the answer must be 72. To makes sure, note that $\sqrt[3]{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9} \approx 71.327$, so P is at least 72. And also note that $P = 72$ can be realized by splitting{1, 8, 9}, {2, 5, 7}, {3, 4, 6}.
18. [D] Say x red chips are placed in the bag, y red chips are placed in the box. Then there would be $10-x-y$ red chips placed in the bowl. Now, x and y are integers such that $0 \le x \le 6$, $0 \le y \le 10$, $0 \le 10-x-y \le 9$. This means (x, y) are lattice points in the Cartesian coordinate plane satisfying $0 \le x \le 6$, $0 \le y \le 10$, $x + y \le 10$, and $(x, y) \ne (0,0)$. This means all lattice points on the trapezoid with $(0,0)$, $(0,10)$, $(6,4)$, $(6,0)$ as the four corners, except the lattice points $(0,0)$. There are therefore a total of $(11+10+9+8+7+6+5)-1 = 55$ lattice points.
19. [D] ΔGEB and ΔGCD are similar. Moreover, $EB : CD = 1:2$. Therefore the ratio of the length of the altitude to BE in ΔGEB to the length of the altitude to DC in ΔGCD is $1:2$. The sum of these two lengths is 72, therefore the length of the altitude to BE in ΔGEB is $\frac{1}{3}(72) = 24$.
20. [E] $r^4 + 2r^9 + \cdots + kr^{5k-1} + \cdots = r^4[1 + 2(r^5) + 3(r^5)^2 + \cdots + k(r^5)^{k-1} + \cdots] = \dfrac{r^4}{(1-r^5)^2}$. The last step is because of $1 + 2u + 3u^2 + \cdots + ku^{k-1} + \cdots = \dfrac{1}{(1-u)^2}$. This can be seen by $1 + 2u + 3u^2 + \cdots + ku^{k-1} + \cdots = \dfrac{d}{du}(1 + u + u^2 + u^3 + \cdots) = \dfrac{d}{du}\dfrac{1}{1-u} = \dfrac{1}{(1-u)^2}$, or by observing that $1 + 2u + 3u^2 + \cdots + ku^{k-1} + \cdots = (1 + u + u^2 + \cdots)(1 + u + u^2 + \cdots)$ by multiplying out the right-hand-side and gathering like terms. But $9r^5 + 7r^2 - 9 = 0$, thus $9(1-r^5) = 7r^2$. Therefore $\dfrac{r^4}{(1-r^5)^2} = \dfrac{r^4}{(7r^2/9)^2} = \dfrac{81}{49}$. The answer is $81 + 49 = 130$.

   Note: Since $P(x)$ is increasing with x for $0 \le x < \infty$, with $P(0) = -9$, $P(1) = 7$, therefore r is somewhere between 0 and 1, hence all the infinite series above converge.